

Reading: FPE, Section 3.5

So far, we have been studying second order systems with transfer functions of the form

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} .$$

This transfer function produced a step response $y(t)$, with Laplace Transform $Y(s) = H(s)\frac{1}{s}$.

Note: All of the analysis so far also holds if we consider the transfer function:

$$H(s) = K \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} ,$$

for some constant K . In this case, the step response is simply scaled by K , but the time characteristics (such as rise time, settling time, peak time and overshoot) of the response are not affected.

We will now consider what happens when we add zeros or additional poles to the system.

1 Effect of a Zero on the Step Response

Suppose that we modify the second order transfer function given above by adding a zero at $s = -z$, for some z . The modified transfer function is given by

$$H_z(s) = \frac{(\frac{1}{z}s + 1)\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} .$$

Note that the reason for writing the zero term as $\frac{1}{z}s + 1$ instead of $s + z$ is to maintain a DC gain of 1 for the transfer function (just so that we can compare it to the original transfer function). We can split the above transfer function into the sum of two terms:

$$H_z(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{1}{z}s \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = H(s) + \frac{1}{z}sH(s) ,$$

where $H(s)$ is the transfer function of the original system (without the zero). Denote the Laplace Transform of the step response for this system by $Y_z(s) = H_z(s)\frac{1}{s}$. Using the above decomposition of $H_z(s)$, we obtain

$$Y_z(s) = \left(H(s) + \frac{1}{z}sH(s) \right) \frac{1}{s} = H(s)\frac{1}{s} + \frac{1}{z}sH(s)\frac{1}{s} = Y(s) + \frac{1}{z}sY(s) .$$

Noting that the inverse Laplace Transform of $sY(s)$ is \dot{y} , we obtain

$$y_z(t) = y(t) + \frac{1}{z}\dot{y}(t) .$$

Thus the step response of the second order system with a zero at $s = -z$ is given by the step response of the original system plus a scaled version of the derivative of the step response of the original system. A sample plot for $z > 0$ (i.e., corresponding to the zero being in the OLHP) is shown in Fig. 1.

Figure 1: Step response of second order system with transfer function $H_z(s) = \frac{(\frac{1}{z}s+1)\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, $z > 0$.

Note that as z increases (i.e., as the zero moves further into the left half plane), the term $\frac{1}{z}$ becomes smaller, and thus the contribution of the term $\dot{y}(t)$ decreases (i.e., the step response of this system starts to resemble the step response of the original system). From the above figure, the effect of a LHP zero is to increase the overshoot, decrease the peak time, and decrease the rise time; the settling time is not affected too much. In other words, a LHP zero makes the step response *faster*.

Now consider what happens if z is negative (which corresponds to the zero being in the ORHP). In this case, the derivative $\dot{y}(t)$ is actually subtracted from $y(t)$ to produce $y_z(t)$. A sample plot is shown in Fig. 2. Note that the response can actually go in the *opposite* direction before rising to 1. This phenomenon is called **undershoot**.

Figure 2: Step response of second order system with transfer function $H_z(s) = \frac{(\frac{1}{z}s+1)\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, $z < 0$.

Zeros in the right half plane are called **nonminimum phase** – the reason for this terminology will become clear later in the course. The effect of a RHP zero is to slow down the system, and perhaps introduce undershoot.

It is also worth noting that if $z = 0$, the above analysis no longer holds directly. Adding a zero at $s = 0$ produces the transfer function $H_z(s) = sH(s)$, and the step response of this system is purely the derivative of the step response of the original system. However, the steady state value of this step response is zero, not 1 (note that this agrees with the DC gain of the new system).

2 Effect of Poles on the Step Response

A general n -th order system will have n poles in the complex plane. As we have already discussed, if any of these poles are in the RHP, the step response will be unstable (i.e., it will go to infinity). Consider the case of all poles being in the OLHP; a sample distribution is shown in Fig. 3.

Figure 3: Sample distribution of poles in OLHP, with dominant poles encircled.

From the partial fraction expansion, we know that a pole $-p$ contributes a term of the form e^{-pt} to the step response. If p is large, this exponential term dies out quickly. Suppose that set of poles for a system can be divided into a cluster of poles that are closest to the origin, and another cluster that are very far away in comparison (e.g., 5 times further away). The poles that are closest to the origin are called the **dominant** poles of the system. The exponential terms in the step response corresponding to the far away poles will die out very quickly in relation to the exponential terms corresponding to the dominant poles. Thus, the system effectively behaves as a lower order system with only the dominant poles. This is one way to approximate a high order system by a lower order system (such as a first or second order system).

Since each additional pole contributes an additional exponential term that must die out before the system reaches its final value, each additional pole increases the rise time of the system. In other words, *adding a pole to the system makes the step response more sluggish*.

Example. Plot the step response of the system with transfer function

$$H(s) = \frac{4}{(\frac{1}{p}s + 1)(s^2 + 2s + 4)} ,$$

for $p = 0.1, 2, 10, 100$.

Solution.

Summary. Based on our discussions, we can make the following observations:

- Adding a LHP zero to the transfer function makes the step response faster (decreases the rise time and the peak time) and increases the overshoot.
- Adding a RHP zero to the transfer function makes the step response slower, and can make the response undershoot.
- Adding a LHP pole to the transfer function makes the step response slower.
- If the system has a cluster of poles and zeros that are much closer (5 times or more) to the origin than the other poles and zeros, the system can be approximated by a lower order system with only those dominant poles and zeros.